PDE I QUALIFYING EXAM, JANUARY 2023

Instructions.

- This is exam is closed-book – no outside materials allowed.
- Please write your solutions neatly and legibly.
- If in doubt, provide more, rather than less, details of proofs or computations. You can apply known theorems and facts from class or the book, but be sure to state the name as best you can (eg, "maximum principle," or "representation formula for solutions to the heat equation.")
- To help graders keep track of your solutions:
  - Write your solution to each question on a separate sheet of paper. (You may use several sheets per question.)
  - Label each sheet with your qual exam id, the question number, sheet number, and (at the end of the exam) the total number of sheets.
  - Do not write on the back of sheets.
  - Paperclip solution sheets together at the end of the exam.

Helpful formulas.

- Fundamental solution of Laplace's equation in $\mathbb{R}^n$:
$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| \text{ for } n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} \text{ for } n \geq 3. \end{cases}$$
- Fundamental solution of the heat equation in $\mathbb{R}^n$:
$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} \text{ for } x \in \mathbb{R}^n \text{ and } t > 0 \\ 0 \text{ for } x \in \mathbb{R}^n \text{ and } t < 0. \end{cases}$$
- Representation formula for solutions of solution of wave equation in 1 dimension with initial data $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$:
$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$
- Characteristic system for $F(Du,u,x) = 0$:
$$\begin{cases} \dot{p} = -D_xF - D_zF \cdot p, \\ \dot{z} = D_pF \cdot p, \\ \dot{x} = D_pF. \end{cases}$$
- Hopf-Lax formula for solution of $u_t + H(Du) = 0$ and initial condition g, where $L = H^*$:
$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}.$$
- Rakine-Hugoniot condition for a shock of a solution of $u_t + F(u)_x = 0$ along a curve C with slope σ:
$$[[F(u)]] = \sigma[[u]].$$

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. Denote $F(z) = \int_0^z f(w)\,dw$. Suppose $u \in C^2([-1,1] \times [0,T])$ solves,

$$\begin{cases} u_{tt} - u_{xx} = f(u) \text{ on } (-1,1) \times (0,T), \\ u(x,0) = g, \quad u_t(x,0) = h \text{ for } x \in (-1,1), \\ u(-1,t) = u(1,t) = 0 \text{ for } t \in (0,T). \end{cases}$$

Consider the energy,

$$e(t) = \frac{1}{2} \int_{-1}^{1} |u_t(x,t)|^2 + |u_x(x,t)|^2 - F(u(x,t))\,dx.$$

Show that e is constant in time. Use this to deduce that if $g, h \equiv 0$, then $u \equiv 0$.

(2) Use the method of characteristics to find the solution u to the PDE

$$x_1 u_{x_1} + x_2 u_{x_2} = 2u$$

on the domain $D := \{(x_1,x_2) |\, x_2 > 1\} \subset \mathbb{R}^2$ that satisfies $u(x_1,1) = g(x_1)$, where $g \in C^1(\mathbb{R})$ is given.

(3) Let $U \subset \mathbb{R}^n$ be open, bounded, and have smooth boundary. Denote $U_T = U \times (0,T]$ and let ΓU_T denote the parabolic boundary of U_T. Consider a solution $u \in C^2(\overline{U_T})$ of

$$\begin{cases} u_t - \Delta u + u^3 = 0 \text{ in } U_T, \\ u = 0 \text{ on } \Gamma U_T. \end{cases}$$

Show that $e(t) := \frac{1}{2}\int_U u^2\,dx$ is non-increasing in time and deduce that $u \equiv 0$.

(4) Suppose u is nonnegative and harmonic on $\mathbb{R}^n$.

(a) Fix any $x, y \in \mathbb{R}^n$. For any $r > 0$, denote $R = r + |x - y|$. Prove

$$u(x) \le \frac{|B_R(y)|}{|B_r(x)|} u(y).$$

Here, $B_R(y)$ denotes the ball of radius R centered at y and $|B_R(y)|$ denotes its volume; and similarly for $B_r(x)$. Hint: notice $B_r(x) \subset B_R(y)$.

(b) Use part (a) to prove that u must be constant.

(5) Write down a representation formula for a solution $u(x,t)$ of,

$$\begin{cases} u_t + b \cdot Du - \Delta u = 0 \text{ on } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) \text{ on } \mathbb{R}^n, \end{cases}$$

where $b \in \mathbb{R}^n$ is a constant and $f \in C^2(\mathbb{R}^n)$. Hint: Make the ansatz that u is of the form $u(x,t) = e^{\alpha \cdot x + \beta t} v(x,t)$ for some v. Find α and β so that v solves the heat equation. Then use the representation formula for solutions of the heat equation to deduce the desired expression for u.